Extending strong conceptual completeness via virtual ultracategories

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An overview

How to categorify topological spaces?

lattice of opens	·	interior comonad	·	relational β-modules
topoi	·	ionads (Garner)	·	virtual ultracategories

Relational β-modules = notion equivalent to topological spaces axiomatizing the convergence of ultrafilters instead of opens subsets.

> An ultrafilter μ converges to a point a if all neighborhoods of a are μ-large we write $\mu \succcurlyeq a$.
>
> For $\mu = \delta_b$ principal, we recover the specialization order: $\delta_b \succcurlyeq a$ iff $b \geq a$.

The ultraconvergence relation is an extension of the specialization order, that is strong enough to recover the topology.

An overview

A topological space is compact Hausdorff iff all ultrafilters have exactly one limit point $\rightsquigarrow$ compact Hausdorff spaces $=$ β-algebras.

Lurie 2018: β-algebras categorify into ultracategories.

Compact Hausdorff $=$ β-alg	$\cdot$	rel β-mod $=$ Topological spaces
ultracategories	$\cdot$	virtual ultracategories

Makkai 1987: conceptual completeness

A coherent topos ($=$ coherent FO-theory)
can be recovered from its ultracategory of points ($=$ models).

We extend this result: a topos with enough points can be recovered from its virtual ultracategory of points.

1) Ultracategories

2) Virtual ultracategories

3) The proof

The usual Stone duality

Classical propositional theories = Boolean Algebras.

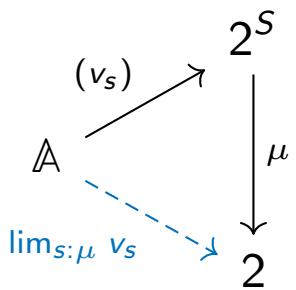
$\mathrm{Mod}(\mathbb{A}) := \mathrm{BoolAlg}(\mathbb{A}, 2)$ has a topology: the Stone topology.

$\mathbb{A}$ can be recovered as the clopen subsets of its space of models:

$$\mathbb{A} \cong \mathrm{TopSp}(\mathrm{Mod}(\mathbb{A}), 2).$$

In particular, $\mathrm{Mod} : \mathrm{BoolAlg}^{\mathrm{op}} \hookrightarrow \mathrm{TopSp}$ is fully faithful.

The Stone topology on $\mathrm{Mod}(\mathbb{A})$ arises from the $\mu \in \mathrm{BoolAlg}(2^S, 2)$



$$\begin{array}{rcl} \mathrm{Mod}(\mathbb{A})^S & \longrightarrow & \mathrm{Mod}(\mathbb{A}) \\ (v_s)_{s:S} & \longmapsto & \lim_{s:\mu} v_s \end{array}$$

$$\beta(S) := \mathrm{BoolAlg}(2^S, 2) = \{\text{ultrafilters on } S\}$$

The $\lim_{s:\mu}(-)$ determine the topology on $\mathrm{Mod}(\mathbb{A})$ by ultraconvergence.

Categorifying to ultracategories

Ultracategories arise by categorifying:

propositional logic	2	boolean algebras
⇕	⇕	⇕
coherent first-order logic	Set	small pretopoi

We want some "categorified Stone topology" to make

$$\begin{array}{rccl} \mathrm{Mod} & : & \mathrm{Pretop}^{\mathrm{op}} & \longrightarrow & \mathrm{CAT} \ + \ ?? \\ & & \mathbb{T} & \longmapsto & \mathrm{Pretop}(\mathbb{T}, \mathrm{Set}) \text{ with the } ??\text{-structure} \end{array}$$

fully faithful.

$\rightsquigarrow$ the ultrastructure on $\mathrm{Mod}(\mathbb{T})$ will play the role of the categorified Stone topology.

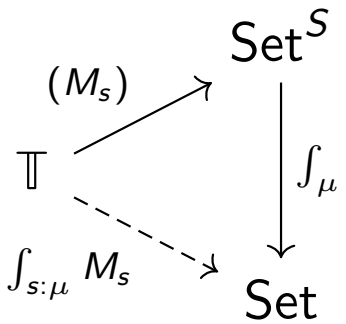
The Stone topology on $\mathrm{Mod}(\mathbb{A})$ comes from $\beta(S) = \mathrm{BoolAlg}(2^S, 2)$
$\rightsquigarrow$ we should look at $\mathrm{Pretop}(\mathrm{Set}^S, \mathrm{Set})$.

Ultraproducts

Łoś's theorem: the ultraproduct functor

$$\begin{array}{rcl} \int_\mu : \mathrm{Set}^S & \longrightarrow & \mathrm{Set} \\ (A_s)_{s:S} & \longmapsto & \int_{s:\mu} A_s := \mathrm{colim}_{\mu(S_0)=1} \prod_{s:S_0} A_s \end{array}$$

is a functor of pretopoi, i.e., $\int_\mu \in \mathrm{Pretop}(\mathrm{Set}^S, \mathrm{Set})$.



induces an operation $\int_\mu : \mathrm{Mod}(\mathbb{T})^S \longrightarrow \mathrm{Mod}(\mathbb{T})$.

ultraproducts = "categorified ultraconverge" on $\mathrm{Mod}(\mathbb{T})$.

Actually, ultraproducts are enough!

$$\mathrm{Pretop}(\mathrm{Set}^S, \mathrm{Set}) \simeq \mathrm{Ind}(\mathcal{UF}^{\mathrm{op}}_{\mathrm{Set}^S}) \qquad \text{Joyal 1971}$$

i.e. a pretopos functor $\mathrm{Set}^S \to \mathrm{Set}$ is a filtered colimit of ultraproducts.

Ultracategories

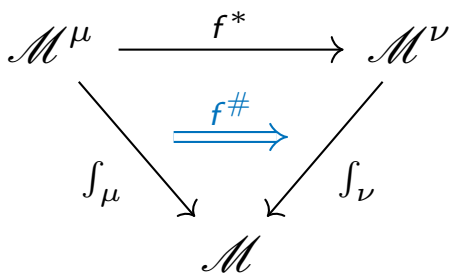
$\mathrm{Mod}(\mathbb{A}) \in \beta$-algebras $=$ compact Hausdorff spaces

$\mathrm{Mod}(\mathbb{T}) \in \beta$-pseudoalgebras $=$ ultracategories

An ultracategory is a category $\mathscr{M}$ together with:

- ultraproduct functors $\int_\mu : \mathscr{M}^\mu \to \mathscr{M}$ $\quad (\mathscr{M}^\mu := \mathrm{colim}_{\mu(S_0)=1} \mathscr{M}^{S_0})$

- functorial reindexing
$f^\# : \int_{s:\mu} A_s \to \int_{t:\nu} A_{f(t)}$, for $f : \nu \to \mu$



- coherent unitor and associators
commuting with the reindexings $\quad \int_\star \xrightarrow{\sim} \mathrm{Id}$, $\int_{\sum_{s:\mu} \nu_s} \xrightarrow{\sim} \int_{s:\mu} \int_{t:\nu_s}$

e.g. Set, Mod($\mathbb{T}$), Compact Hausdorff spaces ($= \beta$-algebras).

Conceptual completeness

Makkai 1987: $\mathbb{T}$ can be recovered from its ultracategory of models:

$$\mathrm{ev} : \mathbb{T} \xrightarrow{\sim} \mathrm{Ult}(\mathrm{Mod}(\mathbb{T}), \mathrm{Set}) \text{ is an equivalence.}$$

$\rightsquigarrow$ conceptual completeness for coherent logic:

$$\mathrm{Mod} : \mathrm{Pretop}^{\mathrm{op}} \hookrightarrow \mathrm{Ult} \text{ is fully faithful.}$$

> Whereas the category $\mathrm{Mod}(\mathbb{T}) \in \mathrm{CAT}$ is not enough to recover $\mathbb{T}$, the ultracategory of models $\mathrm{Mod}(\mathbb{T}) \in \mathrm{Ult}$ is enough to recover $\mathbb{T}$!

How to understand this result?

a model theoritical answer:
ultraproducts are fundamental in the model theory of FO-logic.

a categorical answer: $\mathrm{Pretop}(\mathrm{Set}^S, \mathrm{Set}) \simeq \mathrm{Ind}(\mathcal{UF}^{\mathrm{op}}_{\mathrm{Set}^S})$
ultraproducts are "all" the set operations that preserve FO-logic.

a topological answer:
ultraproducts $\sim$ categorified ultraconvergence.

Reconstruction theorem

Lurie 2018: extends this result to coherent topoi. The functor

$$\mathrm{Pretop}^{\mathrm{op}} \xrightarrow{\mathrm{Sh}} \mathrm{CohTop}$$

is faithful and conservative, but not full.

$$\text{coherent morphisms} \quad \subset \quad \text{geometric morphisms}$$

Accordingly, Lurie introduces a more flexible notion of ultrafunctors.

$$\text{ultrafunctors (Ult)} \quad \subset \quad \text{left-ultrafunctors } (\mathrm{Ult}^{\mathrm{L}})$$

Lurie's reconstruction theorem:

A coherent topos $\mathscr{E}$ can be recovered from its ultracategory of points

$$\mathrm{ev} : \mathscr{E} \xrightarrow{\sim} \mathrm{Ult}^{\mathrm{L}}(\mathfrak{pt}(\mathscr{E}), \mathrm{Set}) \text{ is an equivalence}$$

$\rightsquigarrow$ reconstruction theorem for coherent topoi:

$$\mathfrak{pt} : \mathrm{CohTop} \hookrightarrow \mathrm{Ult}^{\mathrm{L}} \text{ is fully faithful.}$$

Towards virtual ultracategories

Lurie 2018:
$\mathrm{ev} : \mathscr{E} \xrightarrow{\sim} \mathrm{Ult}^{\mathrm{L}}(\mathfrak{pt}(\mathscr{E}), \mathrm{Set})$ is an equivalence for $\mathscr{E}$ coherent topos.

He also shows:
$\mathrm{ev} : \mathscr{E} \xrightarrow{\sim} \mathrm{Ult}^{\mathrm{L}}(\mathfrak{pt}(\mathscr{E}), \mathrm{Set})$ is an equivalence for $\mathscr{E} = \mathrm{Sh}(\text{compact Hausdorff})$.

None of these results contain the other!

We will generalize these both results even further.

We can recover a topos with enough points (= theory)
from the virtual ultracategory structure over its points (= models).

$\rightsquigarrow$ this is a reconstruction theorem for topoi with enough points:

$$\mathfrak{pt} : \mathrm{GTop}_{wep} \hookrightarrow \mathrm{vUlt} \quad \text{is fully faithful.}$$

1) Ultracategories

2) Virtual ultracategories

3) The proof

$\mathscr{E}$ = sheaves over a compact Hausdorff space

X compact Hausdorff $\rightsquigarrow \mathfrak{pt}(X)$ the ultracategory of its points.

$$\begin{array}{ccc} \mathrm{Sh}(X) & \xrightarrow{\sim} & \mathrm{Ult}^{\mathrm{L}}(\mathfrak{pt}(X), \mathrm{Set}) \\ E & \longmapsto & \begin{cases} x \mapsto E_x \\ + \text{ coherent maps } \sigma : E_a \to \int_{x:\mu} E_x \ , \ \text{for } a = \lim \mu. \end{cases} \end{array}$$

A sheaf can be reconstructed from its stalks + some data (the σ's).

We generalize from X compact Hausdorff to T topological space.

We still have the construction,

$$E \in \mathrm{Sh}(T) \longmapsto \begin{cases} x \mapsto E_x \\ + \text{ coherent maps } \sigma_{a,\mu} : E_a \to \int_{x:\mu} E_x \ , \ \text{for } a \preccurlyeq \mu. \end{cases}$$

Can we reconstruct the sheaf E from this data?

$\mathscr{E}$ = sheaves over a topological space

$$E \in \mathrm{Sh}(T) \quad \overset{?}{\simeq} \quad \begin{cases} x \mapsto E_x \\ + \text{ coherent maps } \sigma_{a,\mu} : E_a \to \int_{x:\mu} E_x \ , \ \text{for } a \preccurlyeq \mu. \end{cases}$$

For the reverse construction, the étale space is given by:

$$E := \bigsqcup_{x:T} E_x \ , \ \text{opens} := \text{subsets stable by the } \sigma\text{'s.}$$

To show it is étale, we use the following characterization.

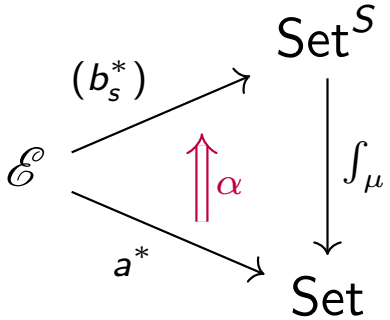
Lemma (S.) : For $p : E \to T$ continuous,

p is étale iff $\forall\ \mu \succcurlyeq p(e),\ \exists!\ \nu \succcurlyeq e,\ p_*(\nu) = \mu$ and this ν is isomorphic to μ via p.

Categorifying ultraconvergence

$a \preccurlyeq (b_s)_{s:\mu}$ means $\forall U$ open , $U \ni a \to (\forall s : \mu,\ U \ni b_s)$

For a, $(b_s)_{s:\mu}$ points of $\mathscr{E}$ topos, we denote by $\alpha : a \prec (b_s)_{s:\mu}$ for



$\alpha \in \mathrm{Nat}_{E:\mathscr{E}}(E_a, \int_{s:\mu} E_{b_s})$

it is a proof-relevant version of "$a \preccurlyeq (b_s)_{s:\mu}$".

$a \prec (b_s)_{s:\mu}$ is an ultraarrow wih codomain an ultrafamily of objects,

these ultraarrows form a generalized multicategorical structure on the points of $\mathscr{E}$ that we denote $\mathfrak{pt}(\mathscr{E}) \in \mathrm{vUlt}$.

$\rightsquigarrow$ the notion of virtual ultracategory.

(generalized multicategories are introduced in Cruttwell and Shulman 2010)

Virtual ultracategories

A virtual ultracategory is a generalized multicategory with arrows having the shape $a \multimap (b_s)_{s:\mu}$. Explicitly:

- a class of objects X,
- a functor $\mathrm{Hom} : X \times \beth(X) \to \mathrm{Set}$ (+ identity, composition...)

  where $\beth(X) \coloneqq \mathcal{UF}^{\mathrm{op}}_{\mathrm{Set}^X}$ is the category of ultrafamilies in X.

  (in particular $\mathrm{id}_a : a \multimap (a)_\star$ induces an arrow $a \multimap (a)_\mu$)

The virtual ultracategory of points of a topos is defined as above

$$\mathfrak{pt} : \mathrm{GTop} \longrightarrow \mathrm{vUlt}$$

with $\mathrm{Ob}(\mathfrak{pt}(\mathscr{E})) \coloneqq \mathrm{pt}(\mathscr{E})$ and $\mathrm{Hom}(a, (b_s)_{s:\mu}) \coloneqq \mathrm{Nat}_{E:\mathscr{E}}(E_a, \int_{s:\mu} E_{b_s})$.

Ultracategories are recovered as the representable virtual ultracategories.

Statement of the theorem

Theorem (S.): Let $\mathscr{E}$ be a topos with enough points, the functor

$$\mathrm{ev} : \mathscr{E} \xrightarrow{\sim} \mathrm{vUlt}(\mathfrak{pt}(\mathscr{E}), \mathrm{Set})$$ is an equivalence.

Some particular cases:

- for $\mathscr{E}$ coherent:
  amounts to Lurie's reconstruction theorem.
- for $\mathscr{E} = \mathrm{Sh}$(topological space):
  amounts to the above lemma characterizing étale maps.
- for $\mathscr{E}$ preasheaf topos:
  follows from Yoneda.

This theorem generalizes, and gives a new proof, to Lurie's result.

1) Ultracategories

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3) The proof

Strategy

We already proved the case $\mathscr{E} = \mathrm{Sh}(\text{topological space})$; to extend from the topological case to the general case, we use representation of topoi by topological groupoids.

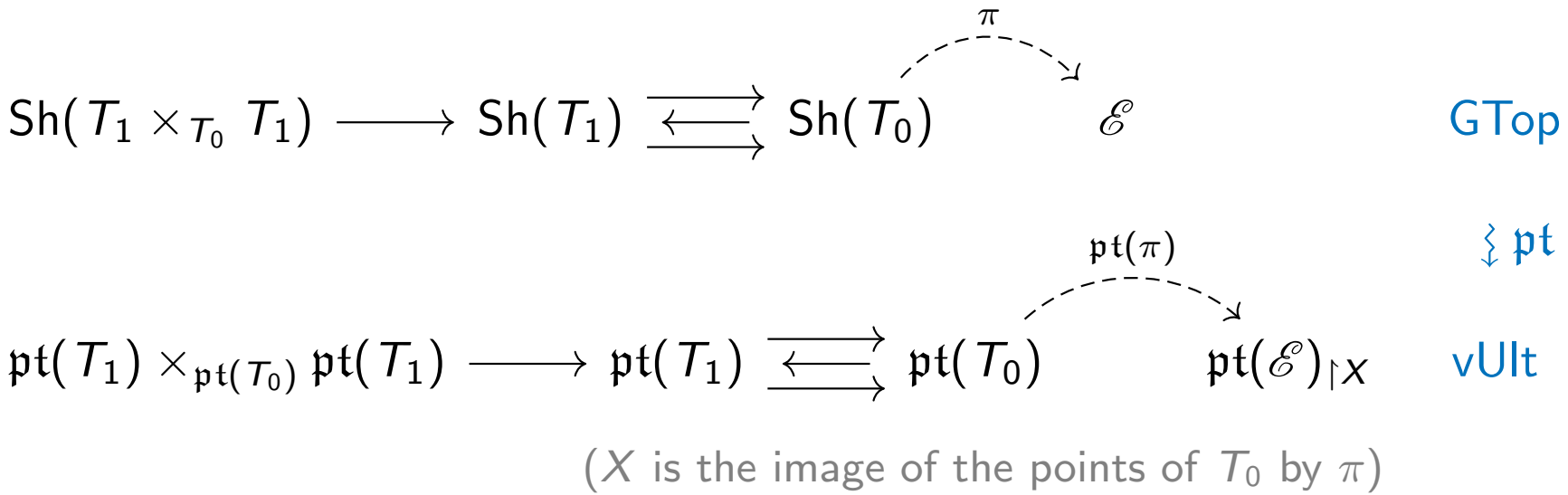
Butz and Moerdijk 1998: any topos with enough points $\mathscr{E}$ can be represented by a topological groupoid $(T_\bullet)$,

i.e. there is a is a universal descent cocone:

$$\mathrm{Sh}(T_1 \times_{T_0} T_1) \xrightarrow{m} \mathrm{Sh}(T_1) \underset{t}{\overset{s}{\underset{\leftarrow u -}{\rightrightarrows}}} \mathrm{Sh}(T_0) \overset{\pi}{\searrow} \mathscr{E}$$

and so, $\mathscr{E} = \mathrm{Sh}_{\mathrm{eq}}(T_\bullet)$ the topos of equivarient sheaves over $(T_\bullet)$.

A problem!



(X is the image of the points of T_0 by π)

$$\mathscr{E} \simeq \mathrm{Cocone}_{\mathrm{desc}}(\mathrm{Sh}(T_\bullet); \mathrm{Set}[\mathbb{O}]) \simeq \mathrm{Sh}_{\mathrm{eq}}(T_\bullet) \simeq \mathrm{Cocone}_{\mathrm{desc}}(\mathfrak{pt}(T_\bullet); \mathrm{Set}) \simeq \mathrm{Sh}_{\mathrm{eq}}(\mathfrak{pt}(T_\bullet)) \overset{??}{\simeq} \mathrm{vUlt}(\mathfrak{pt}(\mathscr{E})_{\restriction X}, \mathrm{Set})$$

$\mathscr{E}$ is the colimit of $(\mathrm{Sh}(T_\bullet))$, we want $\mathfrak{pt}(\mathscr{E}_{\restriction X})$ to be colimit of $(\mathfrak{pt}(T_\bullet))$.

i.e. we want $\mathfrak{pt}(\pi)$ to be effective descent.

Problem: $\mathfrak{pt} : \mathrm{GTop} \to \mathrm{vUlt}$ does not preserve colimits (right adjoint!)

A solution: ample indexings

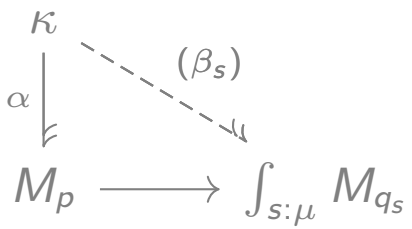
Solution: it works for a well-chosen groupoid!

1) We prove a proposition ensuring a functor of virtual ultracategories to be effective descent.

Proposition: A functor of v-ultracats $F : X \to Y$ surjective on objects and such that any $F(x) \prec (b_s)_{s:\mu}$ can be lift to some $x \prec (y_s)_{s:\mu}$ is effective descent.

2) We consider the groupoid $(T_\bullet^{\mathrm{amp}})$ of small models with an ample indexing, so that $\mathfrak{pt}(\pi)$ satisfies the property of 1.

Definition: An indexing $\alpha : \kappa \twoheadrightarrow M$ is ample if $|\kappa \setminus \mathrm{dom}(\alpha)| = |\kappa|$.



α amply $\rightsquigarrow$ we can extend to surjective (β_s).

3) A theorem of Wrigley 2023 ensures that the ample condition is not too strong, i.e., that $(T_\bullet^{\mathrm{amp}})$ represents $\mathscr{E}$.

The duality

We have shown:

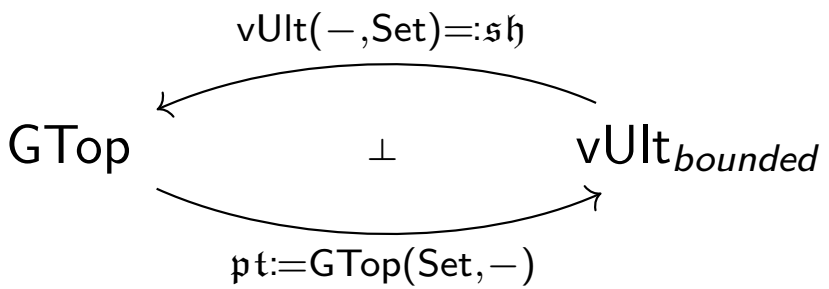
$\mathrm{ev} : \mathscr{E} \xrightarrow{\sim} \mathrm{vUlt}(\mathfrak{pt}(\mathscr{E})_{\restriction X}, \mathrm{Set})$ is an equivalence for $X \subseteq \mathrm{pt}(\mathscr{E})$ small separating.

We can generalize:

$\mathrm{ev} : \mathscr{E} \xrightarrow{\sim} \mathrm{vUlt}(\mathfrak{pt}(\mathscr{E}), \mathrm{Set})$ is an equivalence for any $\mathscr{E}$ with enough points.

Proof: note that $\mathfrak{pt}(\mathscr{E}) \simeq \mathrm{colim}_A \mathfrak{pt}(\mathscr{E})_{\restriction A}$, where A range over small separating sets of points of $\mathscr{E}$, to reduce to the above case. □

We get a pseudoidempotent 2-adjunction

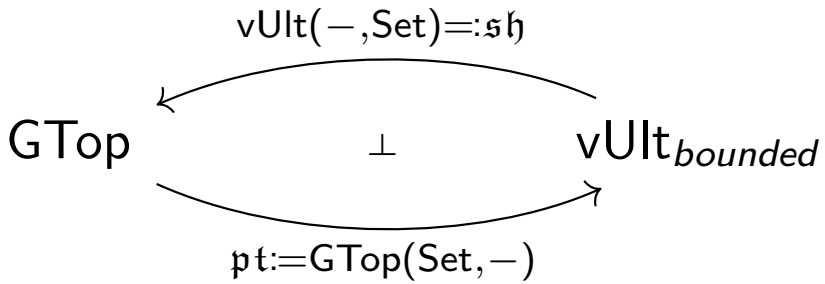


inducing a fully faithful embedding $\mathfrak{pt} : \mathrm{GTop}_{\mathrm{wep}} \hookrightarrow \mathrm{vUlt}$.

Summary

We introduced the notion of virtual ultracategories that categorifies relational β-modules.

We categorified the equivalence between topological spaces and relational β-modules to a pseudoidempotent 2-adjunction.



This adjunction induces a fully faithful embedding $\mathfrak{pt} : \mathsf{GTop}_{\mathsf{wep}} \hookrightarrow \mathsf{vUlt}$ extending Lurie's reconstruction theorem to all topoi with enough points.

Virtual ultracategories fit in the framework of generalized multicategories of Cruttwell and Shulman 2010, and ultracategories are recovered as the representable ones.

References

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Proof of the 0-dimensional case

We prove the non-trivial implication,

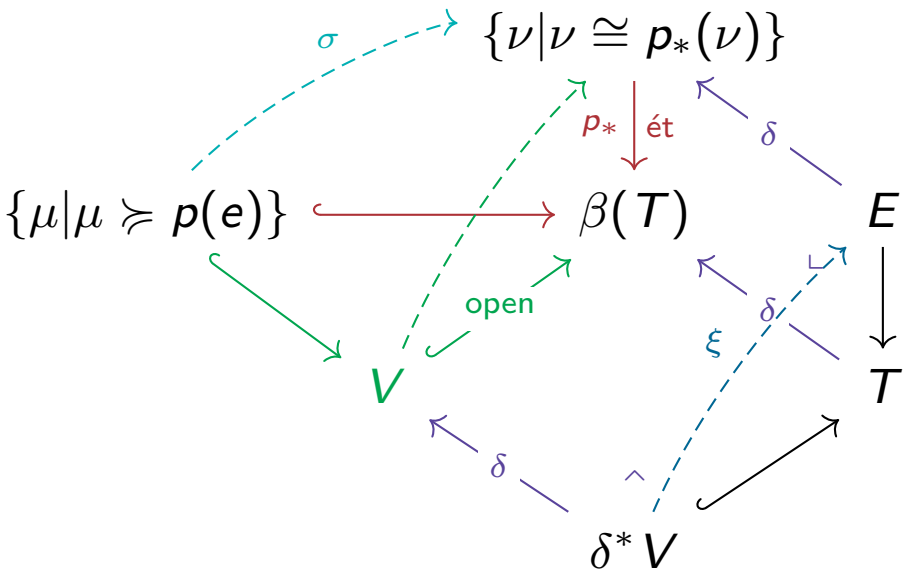
$$p \text{ is étale} \quad \Leftarrow \quad \begin{array}{l} \forall\ \mu \succcurlyeq p(e),\ \exists!\ \nu \succcurlyeq e,\ p_*(\nu) = \mu \\ \text{and this } \nu \text{ is isomorphic to } \mu \text{ via } p. \end{array}$$

- p_* is étale
- the hypothesis gives a lift σ
- the topological fact $(\star)$ gives V
- we pullback along δ

  not continuous!
- $\delta^* T \subseteq T$ is not open

  and ξ not continuous:

  we restrict to $W \subseteq V$.



$$W := \{w \in \delta^* V \mid \forall \mu \succcurlyeq w\ (\xi \text{ is defined on } \mu \text{ and } \xi_*(\mu) \succcurlyeq \xi(w))\}$$

$(\star)$ a section on a compact of a Hausdorff space can be extended to an open. □